

Transformations of some Gauss hypergeometric functions

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Abstract

This paper presents explicit algebraic transformations of some Gauss hypergeometric functions. Specifically, the transformations considered apply to hypergeometric solutions of hypergeometric differential equations with the local exponent differences $1/k, 1/\ell, 1/m$ such that k, ℓ, m are positive integers and $1/k + 1/\ell + 1/m < 1$. All algebraic transformations of these Gauss hypergeometric functions are considered. We show that apart from classical transformations of degree 2, 3, 4, 6 there are several other transformations of degree 6, 8, 9, 10, 12, 18, 24. Besides, we present an algorithm to compute relevant Belyi functions explicitly.

Keywords: Gauss hypergeometric function, algebraic transformation, Belyi function.

1 Introduction

An algebraic transformation of Gauss hypergeometric functions is an identity of the form

$${}_2F_1\left(\begin{array}{c} \tilde{A}, \tilde{B} \\ \tilde{C} \end{array} \middle| x\right) = \theta(x) {}_2F_1\left(\begin{array}{c} A, B \\ C \end{array} \middle| \varphi(x)\right), \quad (1)$$

where $\varphi(x)$ is a rational function of x , and $\theta(x)$ is a product of some powers of rational functions. Here are two examples of quadratic transformations (see [Erd53, AAR99]):

$${}_2F_1\left(\begin{array}{c} a, b \\ \frac{a+b+1}{2} \end{array} \middle| x\right) = {}_2F_1\left(\begin{array}{c} \frac{a}{2}, \frac{b}{2} \\ \frac{a+b+1}{2} \end{array} \middle| 4x(1-x)\right), \quad (2)$$

$${}_2F_1\left(\begin{array}{c} a, \frac{a-b+1}{2} \\ a-b+1 \end{array} \middle| x\right) = \left(1 - \frac{x}{2}\right)^{-a} {}_2F_1\left(\begin{array}{c} \frac{a}{2}, \frac{a+1}{2} \\ \frac{a-b}{2} + 1 \end{array} \middle| \frac{x^2}{(2-x)^2}\right). \quad (3)$$

These identities hold in some neighborhood of $x = 0$ in the complex plane, and can be continued analytically. For example, formula (2) holds for $\operatorname{Re}(x) < 1/2$.

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Recall that the Gauss hypergeometric function ${}_2F_1\left(\begin{smallmatrix} A, B \\ C \end{smallmatrix} \middle| z\right)$ is a solution of the *hypergeometric differential equation*

$$z(1-z)\frac{d^2y(z)}{dz^2} + (C - (A+B+1)z)\frac{dy(z)}{dz} - AB y(z) = 0. \quad (4)$$

This is a Fuchsian equation on the complex projective line \mathbb{P}^1 with 3 regular singular points $z = 0, 1$ and ∞ . The local exponent differences at these points are (up to a sign) $1 - C$, $C - A - B$ and $A - B$ respectively.

Algebraic transformations of Gauss hypergeometric functions usually come from those transformations of hypergeometric equation (4), which have the form

$$z \longmapsto \varphi(x), \quad y(z) \longmapsto Y(x) = \theta(x)y(\varphi(x)), \quad (5)$$

and such that the transformed equation for $Y(x)$ is a hypergeometric equation in the new indeterminate x . Here $\varphi(x)$ and $\theta(x)$ have the same meaning as in formula (1). Geometrically, this is a *pull-back transformation* of equation (4) with respect to the finite covering $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ determined by the rational function $\varphi(x)$. In [Kit03] these transformations are called *RS-transformations*. Recall that a rational function on a Riemann surface is a *Belyi function* [Sha00, Kre03] if it has at most 3 critical values, or equivalently, if the corresponding covering of \mathbb{P}^1 branches only above a set of 3 points. The function $\varphi(x)$ in hypergeometric identities like (1) is usually a Belyi function.

Algebraic transformations of Gauss hypergeometric functions and pull-back transformations of hypergeometric equations are related as follows.

Lemma 1.1 *1. Suppose that pull-back transformation (5) of equation (4) is a hypergeometric equation as well, and that the transformed equation has non-trivial monodromy. Then, possibly after fractional-linear transformations on the projective lines, there is an identity of the form (1) between hypergeometric solutions of the two hypergeometric equations.*

2. Suppose that identity (1) holds in some region of the complex plane. Let $Y(x)$ denote the left-hand side of the identity. If $Y'(x)/Y(x)$ is not a rational function of x , then the transformation (5) converts the hypergeometric equation (4) into a hypergeometric equation for $Y(x)$.

Proof. This is Lemma 2.1 in [Vid04]. □

In this paper, we consider Gauss hypergeometric functions which satisfy hypergeometric equations with local exponent differences $1/k, 1/\ell, 1/m$ such that k, ℓ, m are positive integers and $1/k + 1/\ell + 1/m < 1$. We call these functions *hyperbolic hypergeometric functions*, because they have interesting analytic properties related to the hyperbolic geometry of the complex plane [Yos97, Beu02]. The main purpose of this paper is to describe algebraic transformations of these functions into other hypergeometric functions. Existence

of their non-classical transformations of degree 10, 12 and 24 is shown in [Hod20, Beu02]. A transformation of degree 8 is presented in [Kit03, Section 5]. We give a complete list of possible algebraic transformations of hyperbolic hypergeometric functions. Algebraic transformations of all Gauss hypergeometric functions are classified in [Vid04].

For hyperbolic hypergeometric functions, algebraic transformations always induce pull-back transformations of their hypergeometric equations, and vice versa. Indeed, Kovacic algorithm [Kov86, vdP98] in differential Galois theory implies that the monodromy group of those hypergeometric equations is not trivial, and that they have no solutions $y(z)$ with algebraic logarithmic derivative $y'(z)/y(z)$. Therefore Lemma 1.1 allows no exceptions.

This paper classifies algebraic transformations of hyperbolic hypergeometric functions by finding all pull-back transformations of their hypergeometric equations to other hypergeometric equations. The main problem is to compute suitable coverings $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Possible branching patterns for them are derived in Section 2. We give a general algorithm for computing coverings with prescribed branching pattern in Section 3. Algebraic transformations of hyperbolic hypergeometric functions are listed in Section 4.

2 Possible branching patterns

A general pull-back transformation (5) of a hypergeometric equation is a Fuchsian equation. We are looking for situations when the transformed equation is hypergeometric as well. In this Section we rather look for transformed equations with at most 3 singular points. Since any such Fuchsian equation can be transformed to a hypergeometric equation by fractional-linear transformations, it is appropriate to ignore exact location of singular points for a while. We loosely follow the 5-step classification scheme in [Vid04, Section 3], with $N = 3$, etc.

The requirement that the transformed equation must have at most 3 singular points is restrictive. The covering $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ essentially determines singularities and local exponent differences of the transformed equation. Here are basic general facts which we use (or refer to).

Lemma 2.1 *Let $\varphi : \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$ denote a finite covering of a projective line \mathbb{P}_z^1 with the rational parameter z by a projective line \mathbb{P}_x^1 with the rational parameter x . Let H_1 denote a hypergeometric equation on \mathbb{P}_z^1 , and let H_2 denote the pull-back transformation of H_1 under (5). Let d denote the degree of φ , and let $S \in \mathbb{P}_x^1$, $Q \in \mathbb{P}_z^1$ be points such that $\varphi(S) = Q$.*

1. *If the point Q is non-singular for H_1 , then the point S is non-singular for H_2 only if the covering φ does not branch at S .*
2. *If the point Q is a singular point for H_1 , then the point S is non-singular for H_2 only if the local exponent difference at Q is equal to $1/n$, where n is the branching order of φ at S .*

3. Let Δ denote a set of 3 points on \mathbb{P}_z^1 . If all branching points of φ lie above Δ , then there are exactly $d + 2$ distinct points on \mathbb{P}_x^1 above Δ . Otherwise there are more than $d + 2$ distinct points above Δ .
4. Suppose that the equations H_1 and H_2 are hypergeometric. Let e_1, e_2, e_3 denote the local exponent differences for H_1 , and let e'_1, e'_2, e'_3 denote the local exponent differences for H_2 . Suppose that the local exponent differences are real positive numbers, and that $e_1 + e_2 + e_3 \neq 1$. Then

$$d = \frac{1 - e'_1 - e'_2 - e'_3}{1 - e_1 - e_2 - e_3}. \quad (6)$$

Proof. The first two statements are weaker formulations of parts 2, 3 of [Vid04, Lemma 2.4]. The third statement is part 1 of [Vid04, Lemma 2.5], and the last statement is a weaker formulation of part 2 of [Vid04, Lemma 2.5]; they are consequences of Hurwitz' formula [Har77, Corollary IV.2.4]. \square

Here are restrictions on coverings $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and local exponent differences for algebraic transformations of hyperbolic hypergeometric functions. They are stronger versions of the constraints used in Step 2 of the classification scheme in [Vid04, Section 3].

Lemma 2.2 *Let k, ℓ, m denote positive integers such that*

$$\frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} < 1 \quad \text{and} \quad k \leq \ell \leq m. \quad (7)$$

Let H denote hypergeometric equation (4) such that the local exponent differences are equal to $1/k, 1/\ell, 1/m$. Suppose that pull-back transformation (5) transforms H to a hypergeometric equation. Let d denote the degree of the covering $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

1. *The points $x = 0, 1, \infty$ are actual singularities of the transformed equation, and they lie above the subset $\{0, 1, \infty\}$ of the z -projective line. The covering φ branches only above this subset, so $\varphi(x)$ is a Belyi function.*

2. *The following equality holds:*

$$d - \left\lfloor \frac{d}{k} \right\rfloor - \left\lfloor \frac{d}{\ell} \right\rfloor - \left\lfloor \frac{d}{m} \right\rfloor = 1. \quad (8)$$

3. *The following inequality holds:*

$$d \left(1 - \frac{1}{k} - \frac{1}{\ell} - \frac{1}{m} \right) \leq 1 - \frac{3}{m}. \quad (9)$$

4. *If $m > d$ then $1/d + 1/k + 1/\ell \geq 1$.*

5. If $m \leq d$ then

$$\left(1 - \frac{1}{k} - \frac{1}{\ell}\right)m^2 - 2m + 3 \leq 0 \quad \text{and} \quad \frac{2}{3} \leq \frac{1}{k} + \frac{1}{\ell} < 1. \quad (10)$$

Proof. Let Δ denote the subset $\{0, 1, \infty\}$ of the z -projective line. By part 2 of Lemma 2.1, there are at most $\lfloor d/k \rfloor$, $\lfloor d/\ell \rfloor$, $\lfloor d/m \rfloor$ non-singular points above the z -points with the local exponent differences $1/k$, $1/\ell$, $1/m$ respectively. By part 3 of Lemma 2.1, the number of singular points above Δ is at least

$$d + 2 - \left\lfloor \frac{d}{k} \right\rfloor - \left\lfloor \frac{d}{\ell} \right\rfloor - \left\lfloor \frac{d}{m} \right\rfloor. \quad (11)$$

This number is greater than $2 + d(1 - 1/k - 1/\ell - 1/m)$, so it is at least 3. On the other hand, the transformed equation has at most 3 singular points. Therefore the transformed equation has exactly 3 singular points, expression (11) is equal to 3, and φ does not branch outside Δ by part 1 of Lemma 2.1. Parts 1 and 2 of this Lemma follow.

We rewrite formula (8) as follows:

$$d \left(1 - \frac{1}{k} - \frac{1}{\ell} - \frac{1}{m}\right) + T = 1, \quad (12)$$

where T is the sum of positive local exponent differences at the singular points of the transformed equation. (This is equivalent to (6), with $T = e'_1 + e'_2 + e'_3$.) We have $T \geq 3/m$, which implies inequality (9).

If $m > d$ then we use formula (8) to derive

$$1 = d - \left\lfloor \frac{d}{k} \right\rfloor - \left\lfloor \frac{d}{\ell} \right\rfloor \geq d \left(1 - \frac{1}{k} - \frac{1}{\ell}\right), \quad (13)$$

which gives part 4 of this Lemma. If $m \geq d$, we derive the first inequality in (10) after replacing d by m in (9). We have $1/k + 1/\ell < 1$ from (7). The quadratic expression in m achieves non-positive values only if $1/k + 1/\ell \geq 2/3$ (consider the discriminant). \square

These restrictions essentially give a finite list of possibilities for the integer tuple (k, ℓ, m, d) . Indeed, inequality (9) bounds d once k, ℓ, m are fixed, and part 5 of Lemma 2.2 gives finitely many possibilities for the triple (k, ℓ, m) . Only when $m > d$ we formally have infinitely many possibilities; but then we expect to arrive at specializations of algebraic transformations with unrestricted parameters. The inequalities give the following possibilities:

$$(2, \ell, m, 2), \quad (2, 3, m, 3\dots6), \quad (2, 4, m, 4), \quad (3, 3, m, 3), \quad (2, 3, 7, 7\dots24), \\ (2, 3, 8, 8\dots15), \quad (2, 3, 9, 9\dots12), \quad (2, 3, 10, 10), \quad (2, 4, 5, 5\dots8), \quad (2, 4, 6, 6).$$

Here d is sometimes represented by an integer interval of possible values, and the unevaluated parameters ℓ, m can be large enough integers. Formula (8) rejects some of these possibilities.

Local exponent differences		Degree d	Covering composition	Coxeter decomposition
$(1/k, 1/\ell, 1/m)$	above			
$(1/2, 1/\ell, 1/m)$	$(1/\ell, 1/\ell, 2/m)$	2	indecomposable	yes
$(1/2, 1/3, 1/m)$	$(1/2, 1/m, 2/m)$	3	indecomposable	yes
$(1/2, 1/3, 1/m)$	$(1/3, 1/m, 3/m)$	4	indecomposable	yes
$(1/2, 1/3, 1/m)$	$(1/3, 2/m, 2/m)$	4	no covering	
$(1/2, 1/3, 1/m)$	$(1/m, 1/m, 4/m)$	6	2×3	yes
$(1/2, 1/3, 1/m)$	$(2/m, 2/m, 2/m)$	6	2×3 or 3×2	yes
$(1/2, 1/3, 1/m)$	$(1/m, 2/m, 3/m)$	6	no covering	
$(1/2, 1/4, 1/m)$	$(1/m, 1/m, 2/m)$	4	2×2	yes
$(1/3, 1/3, 1/m)$	$(1/m, 1/m, 1/m)$	3	indecomposable	no

Table 1: Classical transformations of hyperbolic hypergeometric functions

The next step is to produce a list of possible branching patterns. Because of parts 1 and 2 of Lemma 2.2, we have to take the maximal possible number $\lfloor d/k \rfloor$, $\lfloor d/\ell \rfloor$ or $\lfloor d/m \rfloor$ of non-singular points above the 3 singular z -points. The remaining residual branches above $z = 0, 1, \infty$ should coalesce into precisely 3 distinct points. In particular, we have to ignore the cases when there remains less than 3 residual branches. The final list of possible branching patterns is presented in the first three columns of Table 1 (for the cases with $m > d$) and Table 2 (for the cases with $m \leq d$). In Table 1 we ignore fractional-linear transformations, and we drop the condition $\ell \leq m$ for degree 2 transformations. Branching patterns are uniquely determined by the starting local exponent differences $(1/k, 1/\ell, 1/m)$, transformed local exponent differences, and the stated principle to have maximal number of non-singular points above $z = 0, 1, \infty$. For example, branching pattern for the degree 4 transformation of Table 1 between hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/m)$ and $(1/3, 1/m, 3/m)$ can be schematically denoted by $2 + 2 = 3 + 1 = 3 + 1$. This means that all points above the z -point with the local exponent difference $1/2$ have branching order 2, and that there must be a branching point with order 3 and a non-branching point above each of the other two points. For one more example, the degree 9 covering of Table 2 has the branching pattern $2 + 2 + 2 + 2 + 1 = 3 + 3 + 3 = 7 + 1 + 1$. The same notation for branching pattern is used in [Vid04].

Now we have to determine all coverings which have those branching patterns. Given a branching pattern, there is often exactly one covering with that branching pattern up to fractional-linear transformations. But not for any branching pattern a covering exists, and there can be several different coverings with the same branching pattern. Section 3 is devoted to computing coverings with a given branching pattern. First we outline there a straightforward method with undetermined coefficients, which is feasible if $d \leq 6$. Then

Local exponent differences		Degree d	Covering composition	Coxeter decomposition
$(1/k, 1/\ell, 1/m)$	above			
$(1/2, 1/3, 1/7)$	$(1/3, 1/3, 1/7)$	8	indecomposable	no
$(1/2, 1/3, 1/7)$	$(1/2, 1/7, 1/7)$	9	indecomposable	no
$(1/2, 1/3, 1/7)$	$(1/3, 1/7, 2/7)$	10	indecomposable	yes
$(1/2, 1/3, 1/7)$	$(1/7, 1/7, 3/7)$	12	no covering	
$(1/2, 1/3, 1/7)$	$(1/7, 2/7, 2/7)$	12	no covering	
$(1/2, 1/3, 1/7)$	$(1/3, 1/7, 1/7)$	16	no covering	
$(1/2, 1/3, 1/7)$	$(1/7, 1/7, 2/7)$	18	2×9	no
$(1/2, 1/3, 1/7)$	$(1/7, 1/7, 1/7)$	24	3×8	yes
$(1/2, 1/3, 1/8)$	$(1/3, 1/8, 1/8)$	10	indecomposable	no
$(1/2, 1/3, 1/8)$	$(1/4, 1/8, 1/8)$	12	$2 \times 2 \times 3$	yes
$(1/2, 1/3, 1/9)$	$(1/9, 1/9, 1/9)$	12	3×4	no
$(1/2, 1/4, 1/5)$	$(1/4, 1/4, 1/5)$	6	indecomposable	no
$(1/2, 1/4, 1/5)$	$(1/5, 1/5, 1/5)$	8	no covering	

Table 2: Non-classical transformations of hyperbolic hypergeometric functions

we introduce a more appropriate algorithm, which was actually used (within computer algebra system **Maple**) to compute coverings for Table 2. Final information about existing coverings is given in the fourth columns of Table 1 and Table 2. It turns out that for any candidate branching pattern there is at most one covering up to fractional-linear transformations. If $m > d$ (see Table 1), we get the coverings of the classical algebraic transformations due to Gauss, Euler, Kummer, Goursat. If $m \leq d$ (see Table 2), we get new coverings of degree 6, 8, 9, 10, 12, 18, 24. Existence of some of these coverings is shown in [Hod20, Beu02, Kit03]. For both Tables, it was straightforward to figure out possible compositions of small degree coverings and identify them with the unique coverings for suitable branching patterns. Numbers in the multiplicative notation for decomposable coverings mean degrees of constituent coverings, as in [Vid04].

The last step is to determine algebraic transformations of hypergeometric functions with the rational argument determined by a computed covering. The factor $\theta(x)$ in (5) should shift local exponents at potentially non-singular points to the values 0 and 1, and it should shift one local exponent at both $x = 0$ and $x = 1$ to the value 0. A suitable pull-back transformation induces a hypergeometric identity like (1) for each singular x -point S which lies above a singular z -point. To achieve this, one has to move the points S and $\varphi(S)$ to the locations $x = 0$ and $z = 0$ respectively (by fractional-linear transformations), and identify the two solutions with the local exponent 0 and the value 1 at $x = 0$ and $z = 0$ respectively. It is convenient to use Riemann's P -notation for these purposes; see [AAR99, Section 3.9] or [Vid04, Section 2]. Each positioning of $x = 0$ above $z = 0$ gives a

few hypergeometric identities like (1). First of all, we have Euler's and Pfaff's fractional-linear transformations [AAR99, Theorem 2.2.5], which permute other two singular points and their local exponents. Additionally, simultaneous permutation of the local exponents at $x = 0$ and $z = 0$ gives the following hypergeometric identity.

Lemma 2.3 *Suppose that a pull-back transformation induces identity (1) in an open neighborhood of $x = 0$. Then $\varphi(x)^{1-C} \sim Kx^{1-\tilde{C}}$ as $x \rightarrow 0$ for some constant K , and the following identity holds (if both hypergeometric functions are well-defined):*

$${}_2F_1\left(\begin{matrix} 1 + \tilde{A} - \tilde{C}, 1 + \tilde{B} - \tilde{C} \\ 2 - \tilde{C} \end{matrix} \middle| x\right) = \theta(x) \frac{\varphi(x)^{1-C}}{K x^{1-\tilde{C}}} {}_2F_1\left(\begin{matrix} 1 + A - C, 1 + B - C \\ 2 - C \end{matrix} \middle| \varphi(x)\right).$$

Proof. This is Lemma 2.3 in [Vid04]. □

As it turns out, algebraic transformations for Table 1 (i.e., the case $m > d$) are special cases of the classical transformations due to Gauss, Euler, Kummer, Goursat. We give a few instances of these transformations in Section 4. Algebraic transformations for Table 2 (i.e., the case $m \leq d$) are modern, though some of them are predicted in [Hod20]. We present these transformations (up to Euler's and Pfaff's fractional-linear transformations, and Lemma 2.3) in Section 4 as well.

The rest of this Section is devoted to explaining the last columns of Tables 1 and 2. Recall [Yos97, Beu02] that a *Schwarz map* for a hypergeometric equation H_1 is an analytic map from the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ given by a quotient of two solutions of H_1 . If the local exponent differences e_0, e_1, e_∞ of H_1 are real numbers in the interval $[0, 1)$, then the image of a Schwarz map is a curvilinear triangle on the Riemann sphere. Such a triangle is called *Schwarz triangle*. The vertices are images of the 3 singular points, and the angles there are equal to $\pi e_0, \pi e_1, \pi e_\infty$ correspondingly; the sides are circular arcs. Analytic continuation of a Schwarz map follows the *Schwarz reflection principle*: the image of the other half-plane under analytic continuation across $(0, 1), (1, \infty)$ or $(-\infty, 0)$ is a fractional-linear reflection of the Schwarz triangle across the corresponding side of itself.

In our case, the local exponent differences are $1/k, 1/\ell, 1/m$, and $1/k + 1/\ell + 1/m < 1$. The sides of a Schwarz triangle are geodesic curves with respect to a hyperbolic metric on the Riemann sphere, defined on some Poincare disk. Repeated analytic continuation gives a tessellation of the Poincare disk into curvilinear triangles with the angles $\pi/k, \pi/\ell, \pi/m$.

Consider a pull-back transformation of the hypergeometric equation H_1 to a hypergeometric equation H_2 , of degree d . Suppose that its covering $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is defined over \mathbb{R} , and that it branches only above the singular points of H_1 . If $s : \mathbb{H} \rightarrow \mathbb{C}$ is a Schwarz map for H_2 , then a branch of $s \circ \varphi^{-1}$ is a Schwarz map for H_1 . The Schwarz triangles of the d branches of $s \circ \varphi^{-1}$ tessellate the Schwarz triangle of s , like in Figure 1. In this case the degree expression (6) can be interpreted as the quotient of areas of Schwarz triangles for the two hypergeometric equations, in the hyperbolic or spherical metric. Transformations of hypergeometric equations which admit these tessellations are implicitly classified

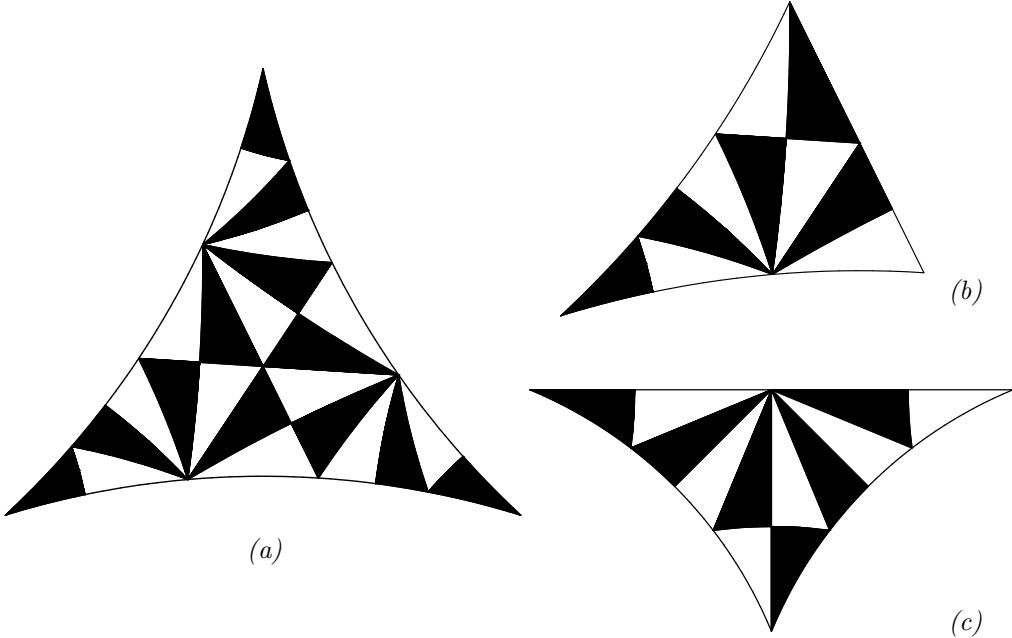


Figure 1: Coxeter decompositions of hyperbolic triangles

in [Hod20, Beu02]. Tessellations of hyperbolic triangles and quadrangles into hyperbolic triangles are classified in [Fel98, BHMM], where they are called *Coxeter decompositions* and *divisible tilings* respectively. The classification in [BHMM] is incomplete; for instance, it misses triangulation (b) in Figure 1. We adopt the terminology of [Fel98].

The last columns of Tables 1 and 2 tell us which transformations of hypergeometric equations with hyperbolic solutions admit Coxeter decompositions of Schwarz triangles. In particular, the three such transformations in Table 2 are anticipated in [Hod20, Beu02]. Their Coxeter decompositions are depicted in Figure 1. All classical transformations except one cubic transformation admit these tessellations.

3 Computation of Belyi functions

Here we consider the problem of computing finite coverings $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of given degree d and with a given branching pattern. We assume that all branching points lie above a set Δ of 3 points, so the desired functions $\varphi(x)$ are Belyi functions. By part 3 of Lemma 2.1, there must be exactly $d+2$ distinct points above Δ . We expect finitely many (or no) solutions to this problem. Algorithm 3.1 of this Section was used to compute transformations implied by Tables 1 and 2. As mentioned, there is at most one solution for branching patterns there.

First we outline a naive method with undetermined coefficients, which is feasible if $d \leq 6$. To fix ideas, consider the branching pattern $2 + 2 + 2 = 3 + 3 = 2 + 2 + 2$ for the sixth entry in Table 1. Up to fractional-linear transformations of the z -line, we may assume that points with these branching orders lie above $z = 1, 0$ and ∞ respectively. We choose the points above $z = \infty$ to be $x = 0, x = 1, x = \infty$. Then the Belyi function should have the form

$$\varphi(x) = \frac{(u_2x^2 + u_1x + u_0)^3}{x^2(x-1)^2}, \quad (14)$$

where u_2, u_1, u_0 are undetermined, and the roots of $u_2x^2 + u_1x + u_0$ are the points above $z = 0$. The branching pattern above $z = 1$ implies that the numerator of $\varphi(x) - 1$ must be a square of a cubic polynomial $P(x)$. This condition gives 7 polynomial equations in the 4 coefficients of $P(x)$ and u_2, u_1, u_0 . These equations can be feasibly solved with assistance of a computer algebra package if d is not large.

To compute Belyi functions more efficiently, we propose to pull-back the differential dz/z with respect to φ , still with undetermined coefficients as in (14). The poles of the pull-backed differential are simple, and they are located at the points above $z = 0$ and $z = \infty$. Its zeroes are the branching points which do not lie above $z = 0$ or $z = \infty$; the multiplicities of the zeroes are 1 less than the corresponding branching orders. In our example, all those branching points must lie above $z = 1$, so they are the roots of $P(x)$. Moreover, they must be simple roots of $P(x)$, so we get this polynomial just by computing the pull-back of dz/z . Explicitly, the pull-back of dz/z is equal to

$$\frac{\varphi'(x)}{\varphi(x)} dx = \frac{2u_2x^3 - (4u_2+u_1)x^2 - (u_1+4u_0)x + 2u_0}{x(x-1)(u_2x^2 + u_1x + u_0)} dx. \quad (15)$$

Let $\tilde{P}(x)$ denote the polynomial in the numerator on the right-hand side; it must be proportional to $P(x)$. Therefore $\varphi(x) - 1$ is proportional to $\tilde{\varphi}(x) := \tilde{P}(x)^2/x^2(x-1)^2$. Further, consider the pull-back of the differential $d(z-1)/(z-1)$:

$$\frac{\tilde{\varphi}'(x)}{\tilde{\varphi}(x)} dx = 4 \frac{u_2x^4 - 2u_2x^3 + (2u_2+u_1+2u_0)x^2 - 2u_0x + u_0}{x(x-1)(2u_2x^3 - (4u_2+u_1)x^2 - (u_1+4u_0)x + 2u_0)} dx. \quad (16)$$

By the same reasoning, the zeroes of this differential are the branching points of φ which do not lie above $z = 1$ or $z = \infty$, with the multiplicities diminished by 1. In our case, those branching points must lie above $z = 0$. Hence the polynomial in the numerator of (16) is proportional to $(u_2x^2 + u_1x + u_0)^2$. This gives easy polynomial equations in u_2, u_1, u_0 . Since we want $u_2u_0 \neq 0$,

$$\frac{u_2^2}{u_2} = -\frac{2u_1u_2}{2u_2} = \frac{u_1^2 + 2u_0u_2}{2u_2 + u_1 + 2u_0} = -\frac{2u_0u_1}{2u_0} = \frac{u_0^2}{u_0}. \quad (17)$$

We solve that $u_2 = -u_1 = u_0$. Therefore $\varphi(x)$ is proportional to $(x^2 - x + 1)^3/x^2(x-1)^2$. The scalar multiple can be found from the condition that $z = 1$ is the third branching locus of $\varphi(x)$. We derive:

$$\varphi(x) = \frac{4}{27} \frac{(x^2 - x + 1)^3}{x^2(x-1)^2}, \quad \varphi(x) - 1 = \frac{(x+1)^2(2x-1)^2(x-2)^2}{27x^2(x-1)^2}.$$

We have solved the problem by hand! The solution is unique up to fractional-linear transformations. Note that there are two different ways to compose coverings of degree 2 and 3 and get a covering with the considered branching pattern; see Table 1 and [Vid04, Section 4]. Up to fractional-linear transformations, those two compositions must give the same covering computed here.

Now we present general Algorithm 3.1 for finding Belyi functions with a given branching pattern. To formulate it more conveniently, we restrict ourselves to branching patterns that are relevant for the purposes of this paper. Note that for all transformations for Table 2 (and almost all transformations for Table 1) there is a z -point with the local exponent difference 1/2. For coverings of these transformations assumption (b) of Algorithm 3.1 holds. If this assumption is dropped, then Step 1 should try to assign the fiber with smallest branching orders to $z = 1$, the function $\tilde{\varphi}(x)$ in Step 3 has a more complicated form, and more undetermined coefficients are needed.

Algorithm 3.1 *Input:* a branching pattern (that is, 3 collections of branching orders) and degree d . We assume:

- (a) the branching orders in the same fiber sum up to d , and there are $d + 2$ branching orders in total;
- (b) one of the 3 collections prescribes only branching orders 2 and at most one unramified (i.e., simple, not branching) point.

Output: All Belyi functions (up to fractional-linear transformations) whose coverings branch only above the set $\Delta = \{0, 1, \infty\}$ with the given branching orders.

Step 1. Prescribe the branching orders mentioned in assumption (b) to the fiber of the point $z = 1$, and prescribe other two collections to the fibers of $z = 0$ and $z = \infty$. Choose the points $x = 0, x = 1, x = \infty$ above Δ in a convenient way: if an unramified point is prescribed above $z = 1$, choose it to be $x = \infty$; see also remarks immediately below. Consider the other points above $z = 0$ and $z = \infty$ as unknown. Accordingly, write $\varphi(x) = K P(x)/Q(x)$, where K is an undetermined constant, and $P(x), Q(x)$ are monic polynomials in the square-free factorized form (following the branching pattern) with some undetermined coefficients.

Step 2. Compute the pull-back $\varphi'(x) dx/\varphi(x)$ of dz/z . Let $R(x)$ be the numerator of $\varphi'(x)/\varphi(x)$. The roots of $R(x)$ are the branching points above $z = 1$.

Step 3. Let $\tilde{\varphi}(x) = R(x)^2/Q(x)$ and compute the rational function $\Phi(x) = \tilde{\varphi}'(x)/\tilde{\varphi}(x)$. The numerator of $\Phi(x)$ has the same roots as the polynomial $P(x)$, but their multiplicity is 1 less than in $P(x)$. This gives a set of algebraic equations in the undetermined coefficients.

Step 4. Solve the algebraic equations obtained in Step 3 by Gröbner basis methods, and find possible pairs of polynomials $P(x), Q(x)$ with the right factorization pattern. For each non-degenerate solution, the constant K in the target $\varphi(x) = K P(x)/Q(x)$ is such that the function $1 - K P(x)/Q(x)$ is proportional to $\tilde{\varphi}(x)$. If necessary, compose the output functions $\varphi(x)$ with suitable fractional-linear transformations to move some x -points (or even z -points) to final desired locations. \square

When applying this algorithm to the entries of Table 2, it is convenient to choose the points $x = 0$, $x = 1$, $x = \infty$ in Step 1 to be the singular points of the transformed hypergeometric equation. In general, a good strategy for Step 1 is to choose points which have different branching orders than the most points in the same fiber. On the other hand, the algorithm can be more effective if we choose points with maximal branching orders as $x = 0$, $x = 1$, $x = \infty$. With this modification, the function $\tilde{\varphi}(x)$ in Step 3 may acquire an extra linear factor. It may be convenient not to make a choice for $x = 1$. Then we would have an extra variable and the algebraic equations would be weighted-homogeneous (respecting to the transformations $x \mapsto \alpha x$). The extra degree of freedom can be used to avoid complicated algebraic numbers.

We note that Step 3 produces enough algebraic equations between the undetermined coefficients, because the restrictions on the polynomials $P(x)$, $Q(x)$, $R(x)$ and the denominator of $\Phi(x)$ determine the desired branching pattern for $\varphi(x)$. Hence the algorithm is correct. As example (17) shows, the set of equations is likely to be overdetermined, which only helps in Gröbner basis computations.

Compared with the naive method described at the beginning of this Section, algebraic equations of Algorithm 3.1 have fewer undetermined coefficients, lower degree, and fewer degenerate (or *parasytic* [Kre03]) solutions. The computations are still tedious, but all coverings of Table 2 were computed using the computer algebra package **Maple** in a matter of hours. There is an article [Sha00] where quadratic differentials are used to characterize some Belyi maps. But [Kre03, Kit03] exploit the naive method.

4 Hypergeometric identities

Here we present our main results. We give all algebraic transformations for Table 2, up to Euler's and Pfaff's fractional-linear transformations and Lemma 2.3. But first we exhibit a few classical transformations.

Relevant instances of quadratic transformations can be obtained by setting $a = 1/2 - 1/\ell - 1/m$, $b = 1/2 - 1/\ell + 1/m$ in formulas (2)–(3). Examples of classical transformations of degree 3 or 4 are:

$${}_2F_1\left(\begin{array}{c} a, \frac{1-a}{3} \\ \frac{4a+5}{6} \end{array} \middle| x\right) = (1-4x)^{-a} {}_2F_1\left(\begin{array}{c} \frac{a}{3}, \frac{a+1}{3} \\ \frac{4a+5}{6} \end{array} \middle| \frac{27x}{(4x-1)^3}\right), \quad (18)$$

$${}_2F_1\left(\begin{array}{c} \frac{4a}{3}, \frac{4a+1}{3} \\ \frac{4a+5}{6} \end{array} \middle| x\right) = (1+8x)^{-a} {}_2F_1\left(\begin{array}{c} \frac{a}{3}, \frac{a+1}{3} \\ \frac{4a+5}{6} \end{array} \middle| \frac{64x(1-x)^3}{(1+8x)^3}\right), \quad (19)$$

$${}_2F_1\left(\begin{array}{c} c, \frac{c+1}{3} \\ \frac{2c+2}{3} \end{array} \middle| x\right) = (1+\omega^2x)^{-c} {}_2F_1\left(\begin{array}{c} \frac{c}{3}, \frac{c+1}{3} \\ \frac{2c+2}{3} \end{array} \middle| \frac{3(2\omega+1)x(x-1)}{(x+\omega)^3}\right). \quad (20)$$

Here ω is a primitive cubic root of unity (so $\omega^2 + \omega + 1 = 0$), $a = 1/4 \pm 3/2m$ and $c = 1/2 \pm 3/2m$. These identities correspond to the indecomposable pull-back coverings

$\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of Table 1. For more complete lists of classical algebraic transformations we refer to [Gou81, Vid04].

Now we present non-classical transformations of hyperbolic hypergeometric functions. Our list of transformations is basically complete, it was computed following the plan of Section 2. If a covering for Table 2 is indecomposable, all corresponding two-term hypergeometric identities can be obtained from the exhibited below, by using Euler's and Pfaff's fractional-linear transformations and Lemma 2.3. If the covering is decomposable, we indicate a composition of hypergeometric identities of smaller degree.

A covering for degree 8 pull-back transformations between hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/7)$ and $(1/3, 1/3, 1/7)$ is given by

$$\varphi_1(x) = \frac{x(x-1)(27x^2 - (723+1392\omega)x - 496+696\omega)^3}{64((6\omega+3)x-8-3\omega)^7}. \quad (21)$$

Here ω satisfies $\omega^2 + \omega + 1 = 0$ as in formula (20). Note that the conjugation $\omega = -1 - \omega$ acts in the same way as a composition with fractional-linear transformation interchanging the points $x = 0$ and $x = 1$. This confirms uniqueness of the covering. The covering is computed in [Kit03] as well. Here are hypergeometric identities:

$${}_2F_1\left(\begin{array}{c} 2/21, 5/21 \\ 2/3 \end{array} \middle| x\right) = (1 - \frac{33+39\omega}{49}x)^{-1/12} {}_2F_1\left(\begin{array}{c} 1/84, 13/84 \\ 2/3 \end{array} \middle| \varphi_1(x)\right), \quad (22)$$

$$\begin{aligned} {}_2F_1\left(\begin{array}{c} 2/21, 3/7 \\ 6/7 \end{array} \middle| x\right) &= (1-x)^{-1/84} \left(1 - \frac{241+464\omega}{9}x - \frac{8(62-87\omega)}{27}x^2\right)^{-1/28} \\ &\times {}_2F_1\left(\begin{array}{c} 1/84, 29/84 \\ 6/7 \end{array} \middle| \frac{1}{\varphi_1(1/x)}\right). \end{aligned} \quad (23)$$

A covering for degree 9 pull-back transformations between hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/7)$ and $(1/2, 1/7, 1/7)$ is given by

$$\varphi_2(x) = \frac{27x(x-1)(49x-31-13\xi)^7}{49(7203x^3 + (9947\xi - 5831)x^2 - (9947\xi + 2009)x + 275 - 87\xi)^3}. \quad (24)$$

Here ξ satisfies $\xi^2 + \xi + 2 = 0$. Hypergeometric identities are:

$$\begin{aligned} {}_2F_1\left(\begin{array}{c} 3/28, 17/28 \\ 6/7 \end{array} \middle| x\right) &= \left(1 + \frac{7(10-29\xi)}{8}x - \frac{343(50-29\xi)}{512}x^2 + \frac{1029(362+87\xi)}{16384}x^3\right)^{-1/28} \\ &\times {}_2F_1\left(\begin{array}{c} 1/84, 29/84 \\ 6/7 \end{array} \middle| \varphi_2(x)\right), \end{aligned} \quad (25)$$

$$\begin{aligned} {}_2F_1\left(\begin{array}{c} 3/28, 1/4 \\ 1/2 \end{array} \middle| x\right) &= \left(1 - \frac{17-29\xi}{21}x - \frac{41+203\xi}{147}x^2 + \frac{275-87\xi}{7203}x^3\right)^{-1/28} \\ &\times {}_2F_1\left(\begin{array}{c} 1/84, 29/84 \\ 1/2 \end{array} \middle| 1 - \varphi_2(1/x)\right). \end{aligned} \quad (26)$$

A covering for degree 10 pull-back transformations between hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/7)$ and $(1/3, 1/7, 2/7)$ is given by

$$\varphi_3(x) = -\frac{x^2(x-1)(49x-81)^7}{4(16807x^3 - 9261x^2 - 13851x + 6561)^3}. \quad (27)$$

Hypergeometric identities are:

$${}_2F_1\left(\begin{matrix} 5/42, 19/42 \\ 5/7 \end{matrix} \middle| x\right) = (1 - \frac{19}{9}x - \frac{343}{243}x^2 + \frac{16807}{6561}x^3)^{-1/28} {}_2F_1\left(\begin{matrix} 1/84, 29/84 \\ 6/7 \end{matrix} \middle| \varphi_3(x)\right), \quad (28)$$

$${}_2F_1\left(\begin{matrix} 5/42, 19/42 \\ 6/7 \end{matrix} \middle| x\right) = (1 - \frac{141}{2}x + \frac{5145}{32}x^2 - \frac{16807}{256}x^3)^{-1/28} {}_2F_1\left(\begin{matrix} 1/84, 29/84 \\ 6/7 \end{matrix} \middle| \varphi_3(1-x)\right), \quad (29)$$

$${}_2F_1\left(\begin{matrix} 5/42, 17/42 \\ 2/3 \end{matrix} \middle| x\right) = (1-x)^{-1/84}(1 - \frac{81}{49}x)^{-1/12} {}_2F_1\left(\begin{matrix} 1/84, 13/84 \\ 2/3 \end{matrix} \middle| \frac{1}{\varphi_3(1/x)}\right). \quad (30)$$

Degree 18 transformations between hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/7)$ and $(1/7, 1/7, 2/7)$ are compositions of degree 9 and degree 2 transformations. The intermediate hypergeometric equation has the local exponent differences $(1/2, 1/7, 1/7)$. To get a hypergeometric identity, one can compose formula (3) with $a = 3/14$, $b = 1/2$ and formula (25).

Degree 24 transformations between hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/7)$ and $(1/7, 1/7, 1/7)$ are compositions of degree 8 and degree 3 transformations. The intermediate hypergeometric equation has the local exponent differences $(1/3, 1/3, 1/7)$. Note that we have here a composition of two pull-back transformations which do not admit a Coxeter decomposition, but the composite transformation does admit a Coxeter decomposition. To get a hypergeometric identity, one can compose formula (20) with $c = 2/7$ and formula (23); see [Vid04, formula (76)].

A covering for degree 10 pull-back transformations between hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/8)$ and $(1/3, 1/8, 1/8)$ is given by

$$\varphi_4(x) = \frac{4x(x-1)(8\beta x+7-4\beta)^8}{(2048\beta x^3 - 3072\beta x^2 - 3264x^2 + 912\beta x + 3264x + 56\beta - 17)^3}. \quad (31)$$

Here β satisfies $\beta^2 + 2 = 0$. Hypergeometric identities are:

$$\begin{aligned} {}_2F_1\left(\begin{matrix} 5/24, 13/24 \\ 7/8 \end{matrix} \middle| x\right) &= \left(1 + \frac{16(4-17\beta)}{9}x - \frac{64(167-136\beta)}{243}x^2 + \frac{2048(112-17\beta)}{6561}x^3\right)^{-1/16} \\ &\times {}_2F_1\left(\begin{matrix} 1/48, 17/48 \\ 7/8 \end{matrix} \middle| \varphi_4(x)\right), \end{aligned} \quad (32)$$

$${}_2F_1\left(\begin{matrix} 5/24, 1/3 \\ 2/3 \end{matrix} \middle| x\right) = (1-x)^{-1/48}\left(1 - \frac{8+7\beta}{16}x\right)^{-1/6} {}_2F_1\left(\begin{matrix} 1/48, 7/48 \\ 2/3 \end{matrix} \middle| \frac{1}{\varphi_4(1/x)}\right). \quad (33)$$

Degree 12 transformations between hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/8)$ and $(1/4, 1/8, 1/8)$ are compositions of a degree 3 transformation and two quadratic transformations. The intermediate hypergeometric equations have the local exponent differences $(1/2, 1/4, 1/8)$ and $(1/2, 1/8, 1/8)$. To get a hypergeometric identity, one can compose formula (3) with $a = 1/4$, $b = 1/2$, formula (2) with $a = 1/8$, $b = 5/8$, and formula (18) with $a = 1/16$.

Degree 12 transformations between hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/9)$ and $(1/9, 1/9, 1/9)$ are compositions of degree 4 and degree 3 transformations. The intermediate hypergeometric equation has the local exponent differences $(1/3, 1/3, 1/9)$. To get a hypergeometric identity, one can compose formula (20) with $c = 1/3$ and formula (19) with $a = 1/12$.

A covering for degree 6 pull-back transformations between hypergeometric equations with the local exponent differences $(1/2, 1/4, 1/5)$ and $(1/4, 1/4, 1/5)$ is given by

$$\varphi_5(x) = \frac{4i x (x-1)(4x-2-11i)^4}{(8x-4+3i)^5}. \quad (34)$$

Hypergeometric identities are:

$${}_2F_1\left(\begin{array}{c} 3/20, 7/20 \\ 3/4 \end{array} \middle| x\right) = \left(1 - \frac{8(4+3i)}{25}x\right)^{-1/8} {}_2F_1\left(\begin{array}{c} 1/40, 9/40 \\ 3/4 \end{array} \middle| \varphi_5(x)\right), \quad (35)$$

$${}_2F_1\left(\begin{array}{c} 3/20, 2/5 \\ 4/5 \end{array} \middle| x\right) = (1-x)^{-1/40}(1 - \frac{2+11i}{4}x)^{-1/10} {}_2F_1\left(\begin{array}{c} 1/40, 11/40 \\ 4/5 \end{array} \middle| \frac{1}{\varphi_5(1/x)}\right). \quad (36)$$

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